ANOTHER PROOF OF WRIGHT'S INEQUALITIES

VLADY RAVELOMANANA

Abstract. We present a short way of proving the inequalities obtained by Wright in [*Journal of Graph Theory*, 4: 393 – 407 (1980)] concerning the number of connected graphs with ℓ edges more than vertices.

1. Preliminaries

For $n \geq 0$ and $-1 \leq \ell \leq \binom{n}{2} - n$, let $c(n,\, n+\ell)$ be the number of connected graphs with n vertices and $n+\ell$ edges. Quantifying $c(n,\, n+\ell)$ represents one of the fundamental tasks in the theory of random graphs. It has been extensively studied since the Erdős-Rényi's paper [3]. The generating functions associated to the numbers $c(n,\, n+\ell)$ are due to Sir E. M. Wright in a series of papers including [11, 12]. He also obtained the asymptotic formula for $c(n,\, n+\ell)$ for every $\ell = o(n^{1/3})$. Using different methods, Bender, Canfield and McKay [1], Pittel and Wormald [8] and van der Hofstad and Spencer [9] were able to determine the asymptotic value of $c(n,\, n+\ell)$ for all ranges of n and ℓ.

For $\ell \geq -1$, let W_ℓ be the exponential generating function (EGF, for short) of the family of connected graphs with n vertices and $n+\ell$ edges. Thus, $W_\ell(z) = \sum_{n=0}^{\infty} c(n,\, n+\ell)\frac{z^n}{n!}$. Let $T(z)$ be the EGF of the Cayley's rooted labeled trees. It is well known that $T(z) = z\, e^{T(z)} = \sum_{n\geq 1} n^{n-1}\frac{z^n}{n!}$ (see for example [4, 5]). Among other results, Wright proved that the functions $W_\ell(z)$, $\ell \geq -1$, can be expressed in terms of $T(z)$. Such results allowed penetrating and precise analysis when studying random graphs processes as it has been shown for example in the giant paper [5]. Throughout the rest of this note, all formal power series are univariate. Therefore, for sake of simplicity we will often omit the variable z so that $T \equiv T(z)$, $W_i \equiv W_i(z)$ and so on.

We need the following notations.

Definition. If A and B are two formal power series such that for all $n \geq 0$ we have $[z^n]\, A(z) \leq [z^n]\, B(z)$ then we denote this relation $A \preceq B$ or $A(z) \preceq B(z)$.

The aim of this note is to provide an alternative and generating function based proof of the inequalities obtained by Sir Wright in [12] (in particular, he used numerous intermediate lemmas). More precisely, Wright obtained the following.

Theorem (Wright 1980). *Let $b_1 = \frac{5}{24}$ and $c_1 = \frac{19}{24}$. Define recursively b_ℓ and c_ℓ by*

$$2(\ell+1)b_{\ell+1} = 3\ell(\ell+1)b_\ell + 3\sum_{t=1}^{\ell-1} t(\ell-t)b_t b_{\ell-t}\,, \qquad (\ell \geq 1) \tag{1}$$

and

$$\begin{aligned} 2(3\ell+2)c_{\ell+1} &= 8(\ell+1)b_{\ell+1} + 3\ell b_\ell + (3\ell+2)(3\ell-1)c_\ell \\ &+ 6\sum_{t=1}^{\ell-1} t(3\ell-3t-1)b_t c_{\ell-t}\,, \qquad (\ell \geq 1) \quad . \end{aligned} \tag{2}$$

Then, for all $\ell \geq 1$

$$\frac{b_\ell}{(1-T(z))^{3\ell}} - \frac{c_\ell}{(1-T(z))^{3\ell-1}} \preceq W_\ell(z) \preceq \frac{b_\ell}{(1-T(z))^{3\ell}} \quad . \tag{3}$$

(3) is known as *Wright's inequalities* and such results has been extremely useful in the enumerative study of graphs as well as in the theory of random graphs [2, 5, 6, 7, 10].

Our proof of (3) is based upon two ingredients:

Fact 1. We know that the EGFs W_ℓ satisfy $W_{-1} = T - \frac{T^2}{2}$, $W_0 = -\frac{1}{2}\log(1-T) - \frac{T}{2} - \frac{T^2}{4}$ and

(4)

$$(1-T)\,\vartheta_z W_{\ell+1} + (\ell+1)\,W_{\ell+1} = \left(\frac{\vartheta_z^2 - 3\vartheta_z}{2} - \ell\right) W_\ell + \frac{1}{2}\sum_{k=0}^{\ell} (\vartheta_z W_k)\,(\vartheta_z W_{\ell-k})\ , \quad (\ell \geq 0)\,,$$

where $T = T(z)$, $W_k = W_k(z)$ and $\vartheta_z = z\frac{\partial}{\partial z}$ corresponds to marking a vertex (such combinatorial operator consists to choose a vertex among the others). For the combinatorial sense of (4), we refer the reader to [1, 5] or [11].

Fact 2. Let A and B be two formal power series and $\ell \in \mathbb{N}$. If $(1-T)\,\vartheta_z A + (\ell+1)\,A \preceq (1-T)\,\vartheta_z B + (\ell+1)\,B$ then $A \preceq B$.

To prove Fact 2, fix $\ell \geq 0$. We write

$$B(z) - A(z) = \sum_{n=0}^{\infty} (b_n - a_n)\frac{z^n}{n!} \quad \text{and} \quad \forall n,\ c_n = b_n - a_n\,. \tag{5}$$

Suppose that $(1-T)\,\vartheta_z A + (\ell+1)\,A \preceq (1-T)\,\vartheta_z B + (\ell+1)\,B$. We then have

$$\begin{aligned} &n!\,[z^n]\,((1-T(z))\,\vartheta_z\,(B(z)-A(z)) + (\ell+1)\,(B(z)-A(z))) = \\ &(n+\ell+1)c_n - \sum_{k=1}^{n} \binom{n}{k} k^{k-1}(n-k)c_{n-k} \geq 0\,. \end{aligned} \tag{6}$$

It is now easily seen that $\forall n, c_n \geq 0$. Therefore, $A \preceq B$.

Our proof of (3) is divided into two parts each of each are given in the next Sections.

2. Proof of $W_\ell \preceq \frac{b_\ell}{(1-T)^{3\ell}}$

Define the family $(\overline{W}_\ell)_{\ell\geq 0}$ as $\overline{W}_0 = -\frac{1}{2}\log(1-T)$ and for $\ell \in \mathbb{N}^\star$, $\overline{W}_\ell = \frac{b_\ell}{(1-T)^{3\ell}}$. Observe that we have $W_0 \preceq \overline{W}_0$ and $W_1 \preceq \overline{W}_1$ has been proved in [12]. Now, we can proceed by induction. Suppose that for $2 \leq i \leq \ell$, $W_i \preceq \overline{W}_i = \frac{b_i}{(1-T)^{3i}}$ and let us prove that $W_{\ell+1} \preceq \overline{W}_{\ell+1} = \frac{b_{\ell+1}}{(1-T)^{3\ell+3}}$. Simple calculations show that

$$(7)\qquad \left(\frac{\vartheta_z^2-\vartheta_z}{2}\right)\overline{W}_\ell \;\preceq\; \frac{\vartheta_z^2}{2}\overline{W}_\ell \;\preceq\; \frac{3\ell(3\ell+2)}{2}\,\frac{b_\ell}{(1-T)^{3\ell+4}} - \frac{3\ell(3\ell+2)}{2}\,\frac{b_\ell}{(1-T)^{3\ell+3}}\,,$$

$$(8)\qquad \left(\vartheta_z\overline{W}_0\right)\,\left(\vartheta_z\overline{W}_\ell\right) \;\preceq\; \frac{3\ell b_\ell}{2}\,\frac{b_\ell}{(1-T)^{3\ell+4}} - \frac{3\ell b_\ell}{2}\,\frac{b_\ell}{(1-T)^{3\ell+3}} \quad \text{and}$$

$$(9)\qquad \frac{1}{2}\sum_{p=1}^{\ell-1}\left(\vartheta_z\overline{W}_p\right)\left(\vartheta_z\overline{W}_{\ell-p}\right) \;\preceq\; \frac{1}{2}\left(\sum_{p=1}^{\ell-1}9p(\ell-p)b_pb_{\ell-p}\right)\left(\frac{1}{(1-T)^{3\ell+4}}-\frac{1}{(1-T)^{3\ell+3}}\right).$$

Summing (7), (8), (9), using the recurrence (1) and the induction hypothesis, we find that

$$(10)\qquad (1-T)\vartheta_z W_{\ell+1} + (\ell+1)W_{\ell+1} \preceq \frac{3(\ell+1)b_{\ell+1}}{(1-T)^{3\ell+4}} - \frac{3(\ell+1)b_{\ell+1}}{(1-T)^{3\ell+3}}\,.$$

Since

$$(11)\qquad (1-T)\vartheta_z\overline{W}_{\ell+1} + (\ell+1)\overline{W}_{\ell+1} = \frac{3(\ell+1)b_{\ell+1}}{(1-T)^{3\ell+4}} - \frac{2(\ell+1)b_{\ell+1}}{(1-T)^{3\ell+3}}$$

by Fact 2, we have $\overline{W}_{\ell+1} \succeq W_{\ell+1}$.

3. Proof of $\frac{b_\ell}{(1-T)^{3\ell}} - \frac{c_\ell}{(1-T)^{3\ell-1}} \preceq W_\ell$

Define $\underline{W}_0 = W_0$ and for $\ell \in \mathbb{N}^\star$, $\underline{W}_\ell = \frac{b_\ell}{(1-T)^{3\ell}} - \frac{c_\ell}{(1-T)^{3\ell-1}}$. As before, we shall proceed by induction. We have $\underline{W}_0 \preceq W_0$ and

$$(12)\qquad W_1 - \underline{W}_1 = \frac{13}{12\,(1-T)} - \frac{1}{2} - \frac{T}{8} + \frac{T^2}{24} \succeq \frac{13}{12}\left(\frac{1}{(1-T)} - T - 1\right) = \frac{13\,T^2}{12(1-T)} \succeq 0\,.$$

Suppose that for $2 \leq k \leq \ell$, $\underline{W}_k = \frac{b_k}{(1-T)^{3k}} - \frac{c_k}{(1-T)^{3k-1}} \preceq W_k$. We have to prove that $\underline{W}_{\ell+1} = \frac{b_{\ell+1}}{(1-T)^{3\ell+3}} - \frac{c_{\ell+1}}{(1-T)^{3\ell+2}} \preceq W_{\ell+1}$. For this purpose, define $\Psi_{\ell+1}$ as

$$\begin{aligned}\Psi_{\ell+1} = &\left(\frac{\vartheta_z^2-3\vartheta_z}{2}-\ell\right)\,\underline{W}_\ell + (\vartheta_z\,\underline{W}_0)\,(\vartheta_z\,\underline{W}_\ell) + \frac{1}{2}\sum_{k=1}^{\ell-1}\left(\vartheta_z\,\underline{W}_k - \frac{(3\ell-1)c_\ell}{(1-T)^{3\ell}}\right)(\vartheta_z\,\underline{W}_{\ell-k})\\ &- \left(\frac{\alpha_\ell}{(1-T)^{3\ell+2}} + \frac{\beta_\ell}{(1-T)^{3\ell+1}} + \frac{\gamma_\ell}{(1-T)^{3\ell}} + \frac{\delta_\ell}{(1-T)^{3\ell-1}}\right),\end{aligned} \qquad (13)$$

where α_ℓ, β_ℓ, γ_ℓ and δ_ℓ are given by

$$\begin{aligned} \alpha_\ell &= \frac{(7\ell+4)c_{\ell+1}}{2} - 3(\ell+1)b_{\ell+1} - \frac{3}{4}\ell b_\ell + \frac{(3\ell-1)(3\ell+4)}{4}c_\ell \\ &+ \frac{1}{2}\sum_{t=1}^{\ell-1}(3t-1)c_t(3\ell-3t-1)c_{\ell-t}\,, \end{aligned} \tag{14}$$

$$\begin{aligned} \beta_\ell &= -\frac{(3\ell+2)c_{\ell+1}}{2} + 2(\ell+1)b_{\ell+1} - \frac{3}{4}\ell b_\ell - \frac{(3\ell-1)(3\ell+4)}{4}c_\ell \\ &- \frac{1}{2}\sum_{t=1}^{\ell-1}(3t-1)c_t(3\ell-3t-1)c_{\ell-t}\,, \end{aligned} \tag{15}$$

$$\gamma_\ell = \frac{\ell b_\ell}{2} + \frac{(3\ell-1)c_\ell}{2} \qquad \text{and} \qquad \delta_\ell = -\frac{\ell-1}{2}c_\ell \quad . \tag{16}$$

Rewritting the formal power series $\frac{\alpha_\ell}{(1-T)^{3\ell+2}} + \frac{\beta_\ell}{(1-T)^{3\ell+1}} + \frac{\gamma_\ell}{(1-T)^{3\ell}} + \frac{\delta_\ell}{(1-T)^{3\ell-1}}$ as follows

$$\begin{aligned} &\frac{(7\ell+4)/2\,c_{\ell+1} - 3(\ell+1)b_{\ell+1} - 3/4\ell b_\ell}{(1-T)^{3\ell+2}} - \frac{(3\ell+2)/2\,c_{\ell+1} - 2(\ell+1)b_{\ell+1} + 3/4\ell b_\ell}{(1-T)^{3\ell+1}} \\ &+ \quad (3\ell-1)(3\ell+4)c_\ell\left(\frac{1}{(1-T)^{3\ell+2}} - \frac{1}{(1-T)^{3\ell+1}}\right) \\ &+ \quad \frac{2\ell b_\ell}{2(1-T)^{3\ell}} + \left(\frac{(3\ell-1)c_\ell}{2(1-T)^{3\ell}} - \frac{(\ell-1)c_\ell}{2(1-T)^{3\ell-1}}\right)\,, \end{aligned} \tag{17}$$

it is easily seen that if the quantity (coming from the denominators of the 2 first terms of the above equation)

$$(2\ell+1)c_{\ell+1} - (\ell+1)b_{\ell+1} - \frac{3}{2}\ell b_\ell \geq 0 \tag{18}$$

then $\frac{\alpha_\ell}{(1-T)^{3\ell+2}} + \frac{\beta_\ell}{(1-T)^{3\ell+1}} + \frac{\gamma_\ell}{(1-T)^{3\ell}} + \frac{\delta_\ell}{(1-T)^{3\ell-1}} \succeq 0$. (We used $1/(1-T)^a \succeq 1/(1-T)^b$ if $a \geq b$).

Using (1) and (2), after simple algebra we have (18). Therefore by construction, RHS of(4) $\succeq \Psi_{\ell+1}$. After nice cancellations, it yields

$$\Psi_{\ell+1} = \frac{3(\ell+1)b_{\ell+1}}{(1-T)^{3\ell+4}} - \frac{2(\ell+1)b_{\ell+1} + (3\ell+2)c_{\ell+1}}{(1-T)^{3\ell+3}} + \frac{(2\ell+1)c_{\ell+1}}{(1-T)^{3\ell+2}}\,. \tag{19}$$

Remarking that $(1-T)\vartheta_z \underline{W}_{\ell+1} + (\ell+1)\,\underline{W}_{\ell+1} = \Psi_{\ell+1}$, we have completed the proof of $\underline{W}_{\ell+1} \preceq W_{\ell+1}$.

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E-mail address: vlad@lipn.univ-paris13.fr

Vlady RAVELOMANANA, LIPN – UMR 7030, Institut Galilée –Université de Paris-Nord, 99, Avenue J. B. Clément. F 93430 Villetaneuse, France.